Derivative

- **Definition and Notation**

We say that a function f is differentiable at a point a iff:

- $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exist and finite

- if f is differentiable at a, then we denote:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

or

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- f is differentiable at a iff f is differentiable from the left and right at a with $f'_+(a) = f'_-(a)$

- **Interpretation of the derivative**
  - The derivative $f'(a)$ is the slope of the tangent line to $f(x)$ at $x = a$

    Equation of the tangent:

    $(T): y = \underbrace{f'(a)}_{\text{slope}} (x - a) + f(a)$

  - If $(d) \parallel (d') \Rightarrow$ slope d = slope d'

Properties:

Let u and v be two differentiable functions then

1. $c' = 0 \quad (c \in \mathbb{R})$
2. $(cu)' = c(u)'$
3. $(u \pm v)' = u' \pm v'$
4. $\left(\frac{1}{u}\right)' = \frac{-u'}{u^2}$
5. $(u.v)' = u'v + v'u$
6. $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$
7. $(u^n)' = n.u^{n-1}.u'$
8. $(\sqrt{u})' = \frac{u'}{2\sqrt{u}}$
9. $(u \circ v)'(x) = u'(v(x)).v'(x)$ (Chain Rule)

Common Derivatives

- $x' = 1$
- $(\sin x)' = \cos x$
- $\sin(u(x))' = u'(x)\cos x$
- $(\cos x)' = -\sin x$
- $\cos(u(x)') = -u'(x)\sin x$
- $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$
- $x^n = nx^{n-1}$
- $(\ln x)' = \frac{1}{x}$
- $\ln(u(x))' = \frac{u'(x)}{u(x)}$, $u \neq 0$, $u(x) > 0$
- $(e^x)' = e^x$
- $(e^{u(x)})' = u'(x)e^{u(x)}$
- $(\tan x)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x$

Higher Order Derivative

. The 2^{nd} Derivative is denoted $f''(x)$

$f''(x) = (f'(x))'$, i.e. the derivative of the first derivative.

. The n^{th} Derivative is denoted as $f^{(n)}(x)$

$f^{n}(x) = (f^{(n-1)}(x))'$, i.e. the derivative of the $(n-1)^{st}$ derivative.

Rolle's Theorem

The function $f : [a,b] \to \mathbb{R}$ s.t.:

. f continuous on $[a,b]$

. f differentiable on $]a,b[$

. $f(a) = f(b)$

then, $\exists\, c \in\,]a,b[\ /\ f'(c) = 0$ (max, min)

Mean Value Theorem (MVT)

Let $f : [a,b] \to \mathbb{R}$ be a real function s.t:

. f continuous on $[a,b]$

. f differentiable on $]a,b[$

then, $\exists\, c \in\,]a,b[\ /\ f(b) - f(a) = f'(c)(b-a)$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

Usual Functions

1- Inverse Function

↳ Definition

If f is a bijective function (injective and surjective)
the f^{-1} (inverse) exist s.t:

$$f : A \longrightarrow B$$
$$f^{-1} : B \longrightarrow A$$

↳ Derivative of an Inverse Function

Let f be a define, c ... and strictly monotone functio on I, then:
If f is differentiable at $a \in I$
with $f'(a) \neq 0$, then f^{-1} is differentiable at $b = f(a)$ with:

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

2- Trigonometric Inverse Functions

↳ arcsin Function

1) . sin(x) : continuous and strictly increasing

$$f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$
$$x \longrightarrow \sin x$$

. arc sin(x): continuous and strictly increasing

$$f^{-1} : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$x \longrightarrow \arcsin(x)$$

2) Properties:

. $\sin(\arcsin x) = x \quad \forall x \in [-1, 1]$

. $\arcsin(\sin x) = x \quad \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

. $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$. $\arcsin'(u(x)) = \frac{u'(x)}{\sqrt{1-u^2(x)}}$

→ arc cos Function

1) Definition

. $\cos(x)$: continuous and strictly decreasing

$$F : [0, \pi] \to [-1, 1]$$
$$x \to \cos(x)$$

. $\arccos(x)$: continuous and strictly decreasing

$$F^{-1} : [-1, 1] \to [0, \pi]$$
$$x \to \arccos(x)$$

2) Properties:

. $\cos(\arccos(x)) = x \quad \forall x \in [-1, 1]$

. $\arccos(\cos(x)) = x \quad \forall x \in [0, \pi]$

. $\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}$. $\arccos'(x) = -\frac{u'(x)}{\sqrt{1-u^2(x)}}$

↳ arctan Function:

1) Definition

. $\tan(x)$: continuous and strictly increasing

$F :]-\frac{\pi}{2}, \frac{\pi}{2}[\longrightarrow \mathbb{R}$

$x \longrightarrow \tan x$

. $\arctan(x)$: continuous and strictly increasing

$F^{-1} : \mathbb{R} \longrightarrow]-\frac{\pi}{2}, \frac{\pi}{2}[$

$x \longrightarrow \arctan(x)$

2) Properties:

. $\tan(\arctan(x)) = x \quad \forall x \in \mathbb{R}$

. $\arctan(\tan x) = x \quad \forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$

. $\arctan'(x) = \frac{1}{1+x^2}$. $\arctan'(u(x)) = \frac{1}{1+u^2(x)}$

3- Logarithm Function (Ln)

1) Definition and notation

. $\ln(x)$: continuous strictly increasing Function

$F :]0, +\infty[\longrightarrow \mathbb{R}$

$x \longrightarrow \ln x$

. $\ln' x = \frac{1}{x}$ $\ln'(u(x)) = \frac{u'(x)}{u(x)}$

. $\ln 1 = 0$. $\ln e = 1$

2) Properties:

- $\ln(ab) = \ln a + \ln b$
- $\ln \frac{a}{b} = \ln a - \ln b$
- $\ln x^n = n \ln x$
- $\ln \frac{1}{n} = -\ln n$

3) Limits values

- $\lim\limits_{x \to +\infty} \ln x = +\infty$
- $\lim\limits_{x \to 0^+} \ln x = -\infty$
- $\lim\limits_{x \to +\infty} \ln x / x^n = 0$
- $\lim\limits_{x \to 0^+} x^n \ln x = 0$
- $\lim\limits_{\substack{x \to 0 \\ x \neq 0}} \frac{\ln(1+x)}{x} = 1$

4- Exponential Function (e)

1) Definition and notation

- e^x: inverse of $\ln(x)$ => continuous and strictly inc.

$$F^{-1} : \mathbb{R} \longrightarrow \left]0, +\infty\right[$$
$$x \longrightarrow e^x$$

- $\ln e^x = x \quad \forall x \in \mathbb{R}$
- $e^{\ln x} = x \quad \forall x > 0$
- $e^x > 0$
- $e^0 = 1 \quad , e^1 = e$

- $(e^x)' = e^x \quad , (e^{u(x)})' = u'(x)\, e^{u(x)}$

2) Properties

- $e^{a+b} = e^a \cdot e^b$
- $e^a / e^b = e^{a-b}$
- $e^{-x} = 1/e^x$
- $(e^x)^n = e^{nx} \quad \forall\, n \in \mathbb{R}$

3) Limits values

- $\lim_{x \to -\infty} e^x = 0$
- $\lim_{x \to +\infty} e^x = +\infty$
- $\lim_{x \to +\infty} \frac{e^x}{x^n} = +\infty$
- $\lim_{x \to -\infty} |x|^n e^x = 0$

5) Power Function

- $F(x) = x^n$

$$F :]0, +\infty[\longrightarrow \mathbb{R}$$
$$x \longrightarrow x^n = e^{n \ln x}$$

- $(x^n)' = n\, x^{n-1}$

→ Properties:

- $x^{a+b} = x^a \cdot x^b$
- $(x^a)^b = x^{ab}$
- $x^{a-b} = \frac{x^a}{x^b}$

Note: $[f(x)]^{g(x)} = e^{g \ln f(x)}$

6 - Hyperbolic Functions

↳ sh and arg sh

1) Definition and notation

. Sh(x): The sine hyperbolic Function, cont. and. inc.

$F : \mathbb{R} \rightarrow \mathbb{R}$

$x \rightarrow \text{Sh}x$

. $\text{Sh}(x) = \dfrac{e^x - e^{-x}}{2}$. $\text{Sh}(0) = 0$

. $\text{Sh}'(x) = \text{ch}(x)$

. arg Sh(x): inverse of Sh(x), continuous and increasing

$F^{-1} : \mathbb{R} \rightarrow \mathbb{R}$

$x \rightarrow \text{arg}\,\text{sh}\,x$

2) Properties

. $\text{sh}(\text{arg}\,\text{Sh}(x)) = x \quad \forall\, x \in \mathbb{R}$

. $\text{arg}\,\text{sh}(\text{Sh}(x)) = x \quad \forall\, x \in \mathbb{R}$

. $\text{arg}\,\text{Sh}'(x) = \dfrac{1}{\sqrt{1+x^2}}$. $\text{arg}\,\text{sh}'(u(x)) = \dfrac{1}{\sqrt{1+u^2(x)}}$

↳ ch and argch

. ch(x): The cosine hyperbolic Function, continuous and increa.

$F : [0, +\infty[\; \rightarrow [1, +\infty[$

$x \rightarrow \text{ch}x$

. $\text{ch}(x) = \dfrac{e^x + e^{-x}}{2}$. $\text{ch}(0) = 1$

. $\text{ch}'(x) = \text{Sh}(x)$

. argch(x): inverse of ch(x) ; continuous and inc.

F^{-1} : $[1, +\infty[\to [0, +\infty[$

$x \to \text{argch}(x)$

. argch(1) = 0

2) Properties :

. $ch(\text{argch } x) = x \quad \forall \; x \in [1, +\infty[$

. $\text{argch}(ch(x)) = x \quad \forall \; x \in [0, +\infty[$

. $\text{argch}'(x) = \frac{1}{\sqrt{x^2 - 1}}$. $\text{argch}'(u(x)) = \frac{1}{\sqrt{u^2(x) - 1}}$

↳ th and arg th

. th(x) : tangent hyperbolic function, cont. and inc.

$F : \mathbb{R} \to]-1, +1[$

$x \to th(x)$

. $th(x) = \frac{Sh(x)}{Ch(x)}$

. $th'(x) = \frac{ch^2 x - Sh^2 x}{Ch^2 x} = \frac{1}{Ch^2 x} = 1 - th^2 x$

. argth(x) : inverse of th(x) ; cont. and inc.

$F^{-1} :]-1, 1[\to \mathbb{R}$

$x \to \text{argth}(x)$

2) Properties

. $th(\text{argth } x) = x \quad \forall \; x \in \mathbb{R}$

. $\text{argth}(th \; x) = x \quad \forall \; x \in]-1, 1[$

. $\text{argth}' x = \frac{1}{1 - x^2}$. $\text{argth}'(u(x)) = \frac{1}{1 - u^2(x)}$